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THE GL(3) MELLIN TRANSFORM FOR TWISTED NON-CUSPIDAL FORMS OF HIGHER LEVEL

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ABSTRACT

The main result in this paper is the explicit computation of the functional equation satisfied by the GL(3) Mellin transform of a twisted non-cuspidal metaplectic form of non-trivial level. For concreteness, we work with one particular metaplectic form, automorphic under $\Gamma(3)$, although our methods extend without change to any form automorphic with respect to $\Gamma(p)$, p an odd prime. We clearly show the computations one must undertake in order to determine the pole locations of this transform (which depend on the form in question), and carry out those straightforward computations in the case of our one specific form.

This particular form, first considered by Bump and Hoffstein [BH], is the maximal parabolic Eisenstein series on the cubic cover of GL(3) (induced from the theta function on the cubic cover of GL(2)), which has the remarkable property that its Fourier coefficients are essentially Hecke cubic L-series. In joint work with Hoffstein, the authors have applied the main theorems in this paper to compute the average values of these Hecke cubic L-series, a result of great arithmetic interest.

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1. Introduction and statement of the main theorems

Let ϕ denote the automorphic form constructed by Bump and Hoffstein [BH], which has the property that for n cubefree the arithmetic part of the Fourier coefficients of ϕ satisfies

$$a_{1,n} = \frac{1}{2} 3^{\frac{11}{2}} |n|^{2(s-1)} L(\tfrac{3}{2}s-1, \chi_{n^2}) L(\tfrac{3}{2}s-1, \chi_1)^{-1},$$

where χ_{n^2} is the cubic residue character mod n^2. Our motivating application is the problem of obtaining average value estimates for the cubic L-series. We will approach this problem in two ways: (1) by trying to obtain a functional equation which can be exploited to yield information about L-series of the shape

(1.1)
$$\sum_{n \text{ squarefree}} \frac{a_{1,n}}{N(n)^w} = \frac{1}{2} 3^{\frac{11}{2}} L(\tfrac{3}{2}s-1, \chi_1)^{-1} \sum_{n \text{ squarefree}} \frac{|n|^{2(s-1)} L(\tfrac{3}{2}s-1, \chi_{n^2})}{N(n)^w}$$
$$= \frac{1}{2} 3^{\frac{11}{2}} L(\tfrac{3}{2}s-1, \chi_1)^{-1} \sum_{n \text{ squarefree}} \frac{L(\tfrac{3}{2}s-1, \chi_{n^2})}{N(n)^{w-s+1}}$$

and translating analytic information about the behavior of (1.1) in the variable w (in terms of s) back into analytic information about the behavior of the cubic L-series; and (2) by constructing a Mellin transform (of an untwisted Eisenstein series) which can be used to construct a series similar to (1.1) but without the restriction that we sum over only squarefree indices.

In this paper we will make a major step towards the construction of (1.1), and complete the second objective entirely. We will obtain information about the analytic properties of the related Dirichlet series

$$L\left(w, \frac{\mu}{q}\right) = \sum_n \frac{e\left(\frac{\mu n}{q}\right) a_{1,n}}{N(n)^{w-s+1}} = \sum_n \frac{e\left(\frac{\mu n}{q}\right) a_{1,n}}{N(n)^{w'}}, \tag{1.2}$$

where w' is just $w - s + 1$, and where we have supressed the dependence on s in our L-series notation. Our information will be in the shape of an explicit function equation, which reflects the series (1.2) into a sum of L-series of a related metaplectic form evaluated at various cusps. Once this evaluation is complete, a simple weighted sum of the series (1.2) as μ and q vary will produce the desired Dirichlet series.

If we specialize to the untwisted case $\mu = 0, q = 1$, then we also obtain the untwisted Mellin transform mentioned above. This somewhat simpler construction has already been applied by Jeffrey Hoffstein and the authors [FHL] to obtain arithmetic results, cf. Remark 1, below.

The twisted Dirichlet series (1.2) may in fact be realized as a double Mellin transform; this is how we will proceed. In particular, we will prove below that

$$\int_0^\infty\int_0^\infty\int_{\mathbb{C}}\sum_{n\neq 0}\phi_{1,n}\left(\begin{pmatrix}1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}yt & & \\ xyt & t & \\ & & 1\end{pmatrix}\right)y^{2w}t^z dx\frac{dt}{t}\frac{dy}{y} = G(w,z)L(w,\mu/q)$$

where

$$L(w,\mu/q)=\sum_{n\neq 0}\frac{a_{1,n}e(n\mu/q)}{(Nn)^w}$$

and

$$G(w,z)=(2\pi)^{-2w-z-2}\times\frac{\Gamma\left(\frac{2w-a}{2}\right)\Gamma\left(\frac{2w-b}{2}\right)\Gamma\left(\frac{2w-c}{2}\right)\Gamma\left(\frac{2-2w+z-a}{2}\right)\Gamma\left(\frac{2-2w+z-b}{2}\right)\Gamma\left(\frac{2-2w+z-c}{2}\right)}{4\Gamma\left(2-2w+\frac{z}{2}\right)}.$$

(Here, a, b, and c are functions of s, which we compute explicitly in the proof of Proposition 1.) In order to state our main results, we first suppose (without any loss of generality) that μ and q are relatively prime. We then have two main results concerning (1.2). They are subtlely different, according as to whether $q\equiv 0 \bmod 3$ or not. Throughout this paper, whenever we must distinguish between these two cases, we will exhibit (and, where illuminating, prove) two analogous results, using the naming convention "Result nA" when q is congruent to a root of unity mod 3, and "Result nB" when $q\equiv 0 \bmod 3$. We have left the functional equations unspecialized (to the Bump–Hoffstein form), although the pole locations are specialized to this one specific form, cf. Remark 0.

Some notation: We write $\mathcal{O}$ for the ring of integers of $\mathbb{Q}(\sqrt{-3})$. The sum $\sum_n$ denotes a sum over $n\in\lambda^{-5}\mathcal{O}$, and Nn denotes the norm of $n\in\mathcal{O}$. We put $\lambda=\sqrt{-3}$, the different of $\mathcal{O}$, and we set $V=\frac{1}{2}3^{1/2}=\mathrm{volume}(\mathbb{C}/\lambda\mathcal{O})$. The notation $\phi_{n,m}$ and ϕ_m^n for the Fourier coefficients of ϕ, and w_n for elements of the Weyl group, are as in [B], [BH] and [BH2].

Main Theorem A: *Suppose q is equivalent to a root of unity mod 3, and $\overline{q}$ is the root of unity so that $q\overline{q}\equiv 1 \bmod 3$. Without loss of generality, we may assume $\mu\equiv 0 \bmod 3$. Then $L(w,\mu/q)$ converges for w with real part sufficiently large, and has a meromorphic continuation to all w with (possible) poles only at the locations $w=s-1$, $w=s$, and $w=\frac{4}{3}-\frac{1}{2}s$. Recalling the definition of $a_{1,n}$, this means that (1.2) has poles at $w'=0$, $w'=1$, and $w'=\frac{4}{3}-\left(\frac{3}{2}s-1\right)$.*

Further, let $\psi(\tau)$ denote the automorphic form defined by $\psi(\tau) = \tilde{\phi}(w_4\tau)$, and let μ' be defined as in Lemma 4A, below, and ρ and B defined as in the proof of Lemma 6A, below. (In particular, μ' depends on μ in a fixed way, and ρ and B depend on the index a, below, in a fixed way.) Let κ_a denote a cube root of unity, depending on a in a fixed way (it is the image of the Kubota homomorphism on the coset depending on a, cf. Lemma 6A). Finally, let $r = (a, \lambda^3 q)/\lambda^3$. Then we have the functional equation

$$G(w,z)L(w,\mu/q) = \frac{1}{VN(\lambda^5 q^3)} \sum_{a\in\mathcal{O}/\lambda^5 q^2} e\left(-\frac{a\bar{q}\mu' - nB(a,\lambda^3 q)}{\lambda^3 q}\right)\kappa_a \times \int_0^\infty\int_0^\infty \sum_{n\neq 0} \psi^{\rho}_{r,n}\left(\begin{pmatrix} ry/q^3 & & \\ & t/r & \\ & & 1\end{pmatrix}\right) y^{-2w} t^z \frac{dt}{t}\frac{dy}{y}.$$

MAIN THEOREM B: *Suppose $q \equiv 0 \bmod 3$, μ is equivalent to a root of unity mod 3, and $\overline{\mu}$ is the root of unity so that $\mu\overline{\mu} \equiv 1 \bmod 3$. Then $L(w,\mu/q)$ converges for w with real part sufficiently large, and has a meromorphic continuation to all w with (possible) poles only at the locations $w = s-1$ and $w = \frac{4}{3} - \frac{1}{2}s$. Further, let $\psi(\tau)$ denote the automorphic form defined by $\psi(\tau) = \tilde{\phi}(w_2\tau)$, and let μ' be defined as in Lemma 4B, below, and ρ and B defined as in the proof of Lemma 6B, below. (In particular, μ' depends on μ in a fixed way, and ρ and B depend on the index a, below, in a fixed way.) Let κ_a denote a cube root of unity, depending on a in a fixed way (it is the image of the Kubota homomorphism on the coset depending on a, cf. Lemma 6B). Finally, let $r = (a, \lambda^3 q)/\lambda^3$. Then we have the functional equation*

$$G(w,z)L(w,\mu/q) = \frac{1}{VN(\lambda^5 q^3)} \sum_{a\in\mathcal{O}/\lambda^5 q^2} e\left(-\frac{a\overline{\mu}\mu' - nB(a,\lambda^3 q)}{\lambda^3 q}\right)\kappa_a \times \int_0^\infty\int_0^\infty \sum_{n\neq 0} \psi^{\rho}_{r,n}\left(\begin{pmatrix} ry/q^3 & & \\ & t/r & \\ & & 1\end{pmatrix}\right) y^{-2w} t^z \frac{dt}{t}\frac{dy}{y}.$$

Remark 0: Our theorems simplify considerably in the case of a form of level 1. In particular, in this case ψ is automorphic for the full modular group, so if we suppose (for this paragraph only) that ϕ is a cusp form of trivial level automorphic with respect to the full group over the rational integers and that q is prime, we obtain the much simpler (single, since both ψ's are the same!) functional equation

$$G(w)L(w,\mu/q) = q^{1-3w}\sum_{n\in\mathcal{O}} \frac{S(\mu',-n,q)\tilde{a}_{1,n}}{|n|^w}\tilde{G}(w) + q^{-w}\sum_{n\in\mathcal{O}} \frac{\tilde{a}_{q,n}}{|n|^w}\tilde{G}(w),$$

where $\tilde{a}_{1,n}$ are the coefficients of $\tilde{\phi}$, where $\tilde{G}$ is the Mellin transform of the Whittaker function of $\tilde{\phi}$, and where S is the classical Kloostermann sum.

Remark 1: Theorem A, specialized to the case where $\mu = 0$, $q = 0$, and some straightforward analytic number theory yields the following result. This is the first non-trivial estimate of the average values of L-series of higher order residue symbols.

THEOREM ([FHL, Main Theorem]): *There exist non-zero constants $c_1(\rho)$ and $c_2(\rho)$ such that*

$$\sum_{\substack{n \in \frac{1}{\lambda^3}\mathcal{O} \\ \mathbb{N}(n) < X}} L^*(\rho, \overline{\chi}_n) = c_1(\rho) X + c_2(\rho) X^{4/3-\rho} + O\left(X^{\theta(\rho)+\epsilon}\right),$$

where $\theta(\rho)$ is the function

$$\theta(\rho) = \begin{cases} \dfrac{7 + 2\rho - 2\rho^2}{2(3 + 2\rho)}, & \frac{1}{2} \le \Re(\rho) \le 1; \\ \dfrac{(3-\rho)(5-4\rho)}{2(6-4\rho)}, & \frac{1}{3} \le \Re(\rho) \le \frac{1}{2}; \end{cases}$$

and $L^*(\rho, \overline{\chi}_n)$ is the Fourier coefficient $a_{1,n}$ evaluated at $s = \frac{2}{3}(\rho + 1)$.

For a discussion of the interpretation of this theorem, and generalizations, we refer the reader to [FHL].

Remark 2: Main Theorems A and B cover seven of the nine congruence classes mod 3. By constructing a weighted sum of the Dirichlet series twisted by characters falling into these seven congruence classes, we may obtain a series which (in each term) differs from (1.1) by a power of 3, and reflects via the functional equation into a scattered sum of Mellin transforms of related automorphic forms. Theorems A and B may then be used to obtain a result similar to that mentioned in Remark 1, but with an index of summation containing only squarefree indices, cf. [FHL, Remark 1].

2. The functional equation and meromorphic continuation

In this section we establish the functional equation and meromorphic continuation of our desired L-series (1.2). We begin by exhibiting a double Mellin transform which we may evaluate directly, and is equal to our (1.2) times a certain explicit gamma factor.

PROPOSITION 1:

$$\int_0^\infty\int_0^\infty\int_{\mathbb{C}}\sum_{n\neq 0}\phi_{1,n}\left(\begin{pmatrix}1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}yt & & \\ xyt & t & \\ & & 1\end{pmatrix}\right)y^{2w}t^z dx\frac{dt}{t}\frac{dy}{y} = G(w,z)L(w,\mu/q) \tag{2.1}$$

where

$$L(w,\mu/q)=\sum_{n\neq 0}\frac{a_{1,n}e(n\mu/q)}{(Nn)^w}$$

and

$$G(w,z)=(2\pi)^{-2w-z-2} \times\frac{\Gamma\left(\frac{2w-a}{2}\right)\Gamma\left(\frac{2w-b}{2}\right)\Gamma\left(\frac{2w-c}{2}\right)\Gamma\left(\frac{2-2w+z-a}{2}\right)\Gamma\left(\frac{2-2w+z-b}{2}\right)\Gamma\left(\frac{2-2w+z-c}{2}\right)}{4\Gamma\left(2-2w+\frac{z}{2}\right)}.$$

Proof: Write $F(w,z)$ for the left side of the Lemma, and put $W(\tau) = W_{1,1}^{(\nu_1,\nu_2)}(\tau)$. We have

$$\begin{aligned}
&F(w,z)\\
&=\sum_{n\neq 0}\frac{a_{1,n}}{Nn}\\
&\quad\times\int_0^\infty\int_0^\infty\int_{\mathbb{C}}W\left(\begin{pmatrix}n & & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}yt & & \\ xt & t & \\ & & 1\end{pmatrix}\right)y^{2w-2}t^z dx\frac{dt}{t}\frac{dy}{y}\\
&=\sum_{n\neq 0}\frac{a_{1,n}}{Nn}\int_0^\infty\int_0^\infty\int_{\mathbb{C}}W\left(\begin{pmatrix}1 & n\mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}nyt & & \\ xt & t & \\ & & 1\end{pmatrix}\right)y^{2w-2}t^z dx\frac{dt}{t}\frac{dy}{y}\\
&=\sum_{n\neq 0}\frac{a_{1,n}e(n\mu/q)}{(Nn)^w}\int_0^\infty\int_0^\infty\int_{\mathbb{C}}W\begin{pmatrix}yt & & \\ xt & t & \\ & & 1\end{pmatrix}y^{2w-2}t^z dx\frac{dt}{t}\frac{dy}{y}\\
&=L(w,\mu/q)G(w,z),
\end{aligned}$$

for some function $G(w,z)$, the triple transform of the Whittaker function.

We turn now to the evaluation of $G(w,z)$. Put $\Delta = \sqrt{1+|x|^2}$. Then

$$W\begin{pmatrix} yt & & \\ xt & t & \\ & & 1 \end{pmatrix} = W\left(\begin{pmatrix} yt & & \\ xt & t & \\ & & 1 \end{pmatrix}\begin{pmatrix} \Delta^{-1} & \overline{x}\Delta^{-1} & \\ -x\Delta^{-1} & \Delta^{-1} & \\ & & 1 \end{pmatrix}\right)$$

$$= W\left(\begin{pmatrix} 1 & \overline{x}y\Delta^{-2} & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} yt\Delta^{-1} & & \\ & t\Delta & \\ & & 1 \end{pmatrix}\right)$$

$$= e\left(\overline{x}y\Delta^{-2}\right) W\begin{pmatrix} yt\Delta^{-1} & & \\ & t\Delta & \\ & & 1 \end{pmatrix}.$$

Thus

$$G(w,z) = \int_0^\infty \int_0^\infty \int_{\mathbb{C}} W\begin{pmatrix} yt\Delta^{-1} & & \\ & t\Delta & \\ & & 1 \end{pmatrix} e\left(\overline{x}y\Delta^{-2}\right) y^{2w-2}t^z dx \frac{dt}{t}\frac{dy}{y}$$

$$= \int_0^\infty \int_0^\infty W\begin{pmatrix} yt & & \\ & t & \\ & & 1 \end{pmatrix} y^{2w-2}t^z \int_{\mathbb{C}} e(\overline{x}y)\Delta^{4w-z-4} dx \frac{dt}{t}\frac{dy}{y}.$$

Write $I(y, -2w+\frac{1}{2}z+2)$ for the integral on x above. Since

$$e(u+iv) = \exp(4\pi i u)$$

we have

$$I(y,\nu) = \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{e(uy)}{(1+u^2+v^2)^\nu} du dv$$

$$= \int_{-\infty}^\infty \frac{e^{4\pi i u y}}{(1+x^2)^{\nu-\frac{1}{2}}} du \int_{-\infty}^\infty \frac{1}{(1+v^2)^\nu} dv$$

$$= \frac{(2\pi)^\nu y^{\nu-1}}{\Gamma(\nu)} K_{\nu-1}(4\pi y).$$

We used Euler's Beta-integral and [GR, 6.576.4] for the last step (cf. [B, Ch.X]).

This brings us to

$$G(w,z)$$
$$=\frac{(2\pi)^{-2w+\frac{z}{2}+2}}{\Gamma\left(-2w+\frac{z}{2}+2\right)}\int_0^\infty\int_0^\infty W\begin{pmatrix} yt & & \\ & t & \\ & & 1\end{pmatrix} K_{-2w+\frac{z}{2}+1}(4\pi y)y^{\frac{z}{2}-1}t^z\frac{dt}{t}\frac{dy}{y}$$

$$=\frac{(2\pi)^{-2w+\frac{z}{2}+2}}{32\Gamma\left(-2w+\frac{z}{2}+2\right)}\int_0^\infty\int_0^\infty K_{a,b,c}(4\pi t,4\pi y)K_{-2w+\frac{z}{2}+1}(4\pi y)(t^2y)^{\frac{z}{2}+1}\frac{dt}{t}\frac{dy}{y}$$

$$=\frac{2^{-2w+\frac{5z}{2}-4}\pi^{-2w-z-1}}{32\Gamma\left(-2w+\frac{z}{2}+2\right)}\int_0^\infty\int_0^\infty K_{a,b,c}(t,y)K_{-2w+\frac{z}{2}+1}(y)(t^2y)^{\frac{z}{2}+1}\frac{dt}{t}\frac{dy}{y}$$

$$=(2\pi)^{-2w-z-2}$$
$$\times\frac{\Gamma\left(\frac{2w-a}{2}\right)\Gamma\left(\frac{2w-b}{2}\right)\Gamma\left(\frac{2w-c}{2}\right)\Gamma\left(\frac{2-2w+z-a}{2}\right)\Gamma\left(\frac{2-2w+z-b}{2}\right)\Gamma\left(\frac{2-2w+z-c}{2}\right)}{4\Gamma\left(2-2w+\frac{z}{2}\right)}.$$

The last step, and the notation (of a, b, and c), are from (1.2) of [B2]. In the case of a $\mathrm{GL}(3,\mathbb{C})$ Eisentein series with parameters ν_1,ν_2, we have

$$a=-\nu_1-2\nu_2+2,\quad b=-\nu_1+\nu_2,\quad\text{and}\quad c=2\nu_1+\nu_2-2$$

(cf. [BF]). The parameters of the Bump–Hoffstein form are

$$\nu_1=s-4/9\quad\text{and}\quad\nu_2=8/9$$

(see [BH] for details).

Now we exhibit the analytic continuation of $L(w,\mu/q)$. This is accomplished by splitting up the integral (2.1) into pieces we can manage directly. Recall the fundamental relationship:

$$\sum_{n\neq0}\phi_{1,n}(\tau)=\phi_1^0(\tau)-\phi_{1,0}(\tau),$$

where $\phi_{m,n}$ satisfies

$$\phi_{m,n}\left(\begin{pmatrix}1 & x_2 & x_3\\ & 1 & x_1\\ & & 1\end{pmatrix}\tau\right)=e(mx_1+nx_2)\phi_{m,n}(\tau).$$

We may decompose the domain of integration in (2.1) into two pieces, and further rewrite the integrand using the difference above, to obtain

$$\begin{aligned}
&\int_1^\infty\int_0^\infty\int_{\mathbb{C}} \sum_{n\neq 0}\phi_{1,n}\left(\begin{pmatrix}1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}yt & & \\ xyt & t & \\ & & 1\end{pmatrix}\right) y^{2w}t^z\,dx\frac{dt}{t}\frac{dy}{y}\\
&+\int_0^1\int_0^\infty\int_{\mathbb{C}} \phi_1^0\left(\begin{pmatrix}1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}yt & & \\ xyt & t & \\ & & 1\end{pmatrix}\right) y^{2w}t^z\,dx\frac{dt}{t}\frac{dy}{y}\\
&-\int_0^1\int_0^\infty\int_{\mathbb{C}} \phi_{1,0}\begin{pmatrix}yt & & \\ xyt & t & \\ & & 1\end{pmatrix} y^{2w}t^z\,dx\frac{dt}{t}\frac{dy}{y}\\
=&I_1+I_2-I_3,
\end{aligned}$$

say. The integrand in I_1 is of rapid-decay, so the integral is convergent for all values of w. The integrals I_2 and I_3 converge for w sufficiently large because each piece has polynomial growth as $y\to 0$. We need only exhibit the analytic continuation of I_2 and I_3 to get the continuation of $L(w,\mu/q)$. The easier integral is I_3, so we compute it first. The proof is deferred to later.

Proposition 2: *If $r\in\lambda^{-3}\mathcal{O}$, then*

$$\phi_{r,0}(\tau)=\frac{1}{V^2}e(rx_1)y_1^{3-s}y_2^{4-2s}F_1(s)I_1(y_1,s)$$

where

$$\begin{aligned}
I_1(y_1,s)=\int_{\mathbb{C}}\int_{\mathbb{C}} &\frac{\left(|\xi_3|^2+|\xi_2|^2+1\right)^{\frac{1}{2}-\frac{3}{2}s}}{|\xi_2|^2+1}K_{\frac{1}{3}}\left(4\pi|r|y_1\frac{\sqrt{|\xi_3|^2+|\xi_2|^2+1}}{|\xi_2|^2+1}\right)\\
&\times e\left(y_1\frac{-\xi_2\xi_3}{|\xi_2|^2+1}\right)d\xi_{2,3};\\
F_1(s)=\sum_{C\equiv 1(3)}&\sum_{\substack{A,B \bmod C\\ (A,B,C)=1\\ A\equiv B\equiv 0(3)\\ d^2|\lambda^3 rC}}\left(\frac{b}{c}\right)\left(\frac{A}{d}\right)e(rqA/d)\tau(rC/d^2)\frac{d^2}{C^{1+3s}};
\end{aligned}$$

and $\tau(\cdot)$ is the coefficient of the cubic theta function studied by Patterson [P1, P2], whose properties are summarized in (1.15) of [BH]. In particular,

$$\begin{aligned}
&\int_0^1\int_0^\infty\int_{\mathbb{C}}\phi_{1,0}\begin{pmatrix}yt & & \\ xyt & t & \\ & & 1\end{pmatrix}y^{2w}t^z\,dx\frac{dt}{t}\frac{dy}{y}\\
&\qquad=\frac{\pi}{V^2}F_1(s)\frac{1}{(w-s+1)(\frac{z}{2}-2s-3)}\int_0^\infty I_1(t,s)t^{3-s+z}\frac{dt}{t}.
\end{aligned}$$

The last line of Proposition 3 provides the functional equation and meromorphic continuation of I_3. To determine the functional equation and meromorphic continuation of I_2, we require two key steps completely analogous to the two key steps of Bump's formulas on $GL(3,\mathbb{R})$, [B, 8.5] and [B, 8.6]. In particular, we will prove the following two lemmas, whose proofs we defer to the next section. These two lemmas essentially yield Lemma 5, an expansion of the inner integrand in I_2.

LEMMA 3:

$$\int_{\mathbb{C}} \phi_1^0 \left(\begin{pmatrix} 1 & \mu/q & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ x & 1 & \\ & & 1 \end{pmatrix} \tau \right) dx$$
$$= \frac{1}{VN(\lambda^5 q^3)} \cdot \sum_{a\in\mathcal{O}/\lambda^5 q^2} \int_{\mathbb{C}/3q} \int_{\mathbb{C}/3q^2} \phi \left(\begin{pmatrix} 1 & \mu/q & a/\lambda^3 q^2 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ x & 1 & \xi_1 \\ & & 1 \end{pmatrix} \tau \right) e(-\xi_1) dx d\xi_1.$$

LEMMA 4A: *Suppose $(\mu, q) = 1$ with $3|\mu$ and q equivalent to a root of unity mod 3. Let $\overline{q}$ denote the root of unity such that $q\overline{q} \equiv 1$ mod 3. Choose d and μ' so that $3|\mu'$ and $dq\overline{q} - \mu\mu'\overline{q} = 1$. Then,*

$$\phi\left(\begin{pmatrix} 1 & \mu/q & a/\lambda^3 q^2 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} yt & & \\ xyt & t & \xi_1 \\ & & 1 \end{pmatrix} \right)$$
$$= \tilde{\phi}\left(w_4 \begin{pmatrix} 1 & & \mu'/q \\ a\overline{q}/\lambda^3 q & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1/q\overline{q}y & & x/q\overline{q} \\ & t & \xi_1 \\ & & q \end{pmatrix} \right).$$

LEMMA 4B: *Suppose $(\mu, q) = 1$ with $3|q$. Then μ is equivalent to a root of unity mod 3. Let $\overline{\mu}$ denote the root of unity so that $\mu\overline{\mu} \equiv 1$ mod 3. Choose d and μ' so that $3|d$, $\mu' \equiv 1$ mod 3, and $-dq\overline{\mu} + \mu'\mu\overline{\mu} = 1$. Then,*

$$\phi\left(\begin{pmatrix} 1 & \mu/q & a/\lambda^3 q^2 \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} yt & & \\ xyt & t & \xi_1 \\ & & 1 \end{pmatrix} \right)$$
$$= \tilde{\phi}\left(w_2 \begin{pmatrix} 1 & & \mu'/q \\ a\overline{\mu}/\lambda^3 q & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1/q\overline{\mu}y & & x/q\overline{\mu} \\ & t & \xi_1 \\ & & q \end{pmatrix} \right).$$

LEMMA 5A: *Suppose $(\mu, q) = 1$ with $3|\mu$ and q equivalent to a root of unity mod 3. With notation as above, we have*

$$\int_{\mathbb{C}} \phi_1^0 \left(\begin{pmatrix} 1 & \mu/q & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} yt & & \\ xyt & t & \\ & & 1 \end{pmatrix} \right) dx$$
$$= \frac{1}{VN(\lambda^5 q^3)} \sum_{a \in \mathcal{O}/\lambda^5 q^2} e\left(-\frac{a\bar{q}\mu'}{\lambda^3 q}\right)$$
$$\cdot \int_{(\mathbb{C}/3)^2} \tilde{\phi} \left(w_4 \begin{pmatrix} 1 & & x \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ a\bar{q}/\lambda^3 q & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1/y\bar{q}q^2 & & \\ & t/q & \\ & & 1 \end{pmatrix} \right)$$
$$\times e\left(-q\xi_1 + \frac{xa\bar{q}}{\lambda^3}\right) dx d\xi_1.$$

LEMMA 5B: *Suppose $(\mu, q) = 1$ with $3|q$ and μ is equivalent to a root of unity mod 3. With notation as above, we have*

$$\int_{\mathbb{C}} \phi_1^0 \left(\begin{pmatrix} 1 & \mu/q & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} yt & & \\ xyt & t & \\ & & 1 \end{pmatrix} \right) dx$$
$$= \frac{1}{VN(\lambda^5 q^3)} \sum_{a \in \mathcal{O}/\lambda^5 q^2} e\left(-\frac{a\bar{\mu}\mu'}{\lambda^3 q}\right)$$
$$\cdot \int_{(\mathbb{C}/3)^2} \tilde{\phi} \left(w_2 \begin{pmatrix} 1 & & x \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ a\bar{\mu}/\lambda^3 q & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1/y\bar{\mu}q^2 & & \\ & t/q & \\ & & 1 \end{pmatrix} \right)$$
$$\times e\left(-q\xi_1 + \frac{xa\bar{\mu}}{\lambda^3}\right) dx d\xi_1.$$

Remark: The proof of Lemma 5B is completely analogous to that of Lemma 5A. We include below only the proof of Lemma 5A.

Proof of Lemma 5A: Write $I_{\mathbb{C}}$ for the left side of Lemma 5A. Combining Lemmas 3 and 4A gives

$$I_{\mathbb{C}} = \frac{1}{VN(\lambda^5 q^3)} \sum_{a \in \mathcal{O}/\lambda^5 q^2}$$
$$\cdot \int_{\mathbb{C}/3q} \int_{\mathbb{C}/3q^2} \tilde{\phi} \left(w_4 \begin{pmatrix} 1 & & \mu'/q \\ a\bar{q}/\lambda^3 q & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1/q\bar{q}y & & x/q\bar{q} \\ & t & \xi_1 \\ & & q \end{pmatrix} \right) e(-\xi_1) dx d\xi_1.$$

It is easy to verify that

$$\begin{pmatrix} 1 & & \mu'/q \\ a\overline{q}/\lambda^3 q & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1/q\overline{q}y & & x/q\overline{q} \\ & t & \xi_1 \\ & & q \end{pmatrix} =$$

$$\begin{pmatrix} 1 & & \frac{x}{q^2\overline{q}} + \frac{\mu'}{q} \\ & 1 & \frac{\xi_1}{q} + \frac{xa}{\lambda^3 q^3} \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & & \\ a\overline{q}/\lambda^3 q & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1/yq\overline{q} & & \\ & t & \\ & & q \end{pmatrix}.$$

Inserting this into the above equation and making the change of variables $\xi_1 \to q\xi_1 - xa/\lambda^3 q^2$ and $x \to \overline{q}q^2 x - \mu' q\overline{q}$ finishes the proof of Lemma 5.

We now introduce the notation $\psi(\tau)$, which we will use interchangeably for the two automorphic forms defined by $\psi(\tau) = \tilde{\phi}(w_4\tau)$ and $\psi(\tau) = \tilde{\phi}(w_2\tau)$. We use the first definition when discussing Main Theorem A, and the second when discussing Main Theorem B. In particular, we will use the notation $\psi(\tau) = \tilde{\phi}(w_4\tau)$ for the following discussion and what follows, until we have completed the proof of Main Theorem A. We will also restrict ourselves to the case when q is congruent to a root of unity mod 3, and $3|\mu$, until the proof of Main Theorem A is complete.

Since $\Gamma(3)$ is normal in $\Gamma(1)$, ψ is again an automorphic form, and has an expansion into terms of the form

$$\psi^{\rho}_{m,n}\left(\begin{pmatrix} A & B & \\ C & D & \\ & & 1 \end{pmatrix}\tau\right), m \in \lambda^{-3}\mathcal{O}, n \in \lambda^{-5}\mathcal{O}, A, B, C, D \in \mathcal{O}, AD - BC = 1,$$

where ρ depends on A, B, C, D and $\psi^{\rho}_{m,n}(\tau)$ satisfies

$$\psi^{\rho}_{m,n}\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\tau\right) = e(mx_1 + nx_2)\psi^{\rho}_{m,n}(\tau). \tag{2.2}$$

See [BH2] for details.

The next step is to replace ψ by its Fourier expansion. Most of the terms in the Fourier expansion of ψ will not contribute to the integral on the right hand side of Lemma 5. Indeed, the contribution of a given term is just

(2.3)

$$\int_{(\mathbb{C}/3)^2} \psi^{\rho}_{m,n}\left(\begin{pmatrix} A & B & \\ C & D & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & x \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ a\overline{q}/\lambda^3 q & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1/y\overline{q}q^2 & & \\ & t/q & \\ & & 1 \end{pmatrix}\right)$$

$$\times\, e\left(-q\xi_1 + \frac{xa\overline{q}}{\lambda^3}\right) dx d\xi_1.$$

LEMMA 6A: *Under the hypotheses of Lemma 5A, the integral (2.3) is zero except possibly when*

$$m = (a, \lambda^3 q)/\lambda^3,$$
$$C = -a\overline{q}/(a, \lambda^3 q),$$
$$D = \lambda^3 q/(a, \lambda^3 q).$$

In particular,

$$(2.4)\quad \int_{\mathbb{C}} \sum_n \phi_{1,n}\left(\begin{pmatrix} 1 & \mu/q & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} yt & & \\ xyt & t & \\ & & 1 \end{pmatrix}\right) dx$$
$$= \frac{1}{VN(\lambda^5 q^3)} \sum_{a \in \mathcal{O}/\lambda^5 q^2} e\left(-\frac{a\overline{q}\mu' - nB(a, \lambda^3 q)}{\lambda^3 q}\right)\kappa_a \sum_n \psi^\rho_{r,n}\begin{pmatrix} r/yq^3 & & \\ & t/r & \\ & & 1 \end{pmatrix},$$

where $r = (a, \lambda^3 q)/\lambda^3$ *and* $A\lambda^3 q + Ba\overline{q} = (a, \lambda^3 q)$, *and* ρ *depends on* a, q, *and* B.

Proof: First note that

$$\begin{pmatrix} A & B & \\ C & D & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & x \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & Ax + B\xi_1 \\ & 1 & Cx + D\xi_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} A & B & \\ C & D & \\ & & 1 \end{pmatrix}$$

and thus (2.3) is equal to

$$\int_{(\mathbb{C}/3)^2} \psi^\rho_{m,n}\left(\begin{pmatrix} A & B & \\ C & D & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ a\overline{q}/\lambda^3 q & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1/y\overline{q}q^2 & & \\ & t/q & \\ & & 1 \end{pmatrix}\right)$$
$$\times\, e\left((mD - q)\,\xi_1 + \left(mC + \frac{a\overline{q}}{\lambda^3}\right)x\right) dx d\xi_1.$$

The integral above will be zero unless $mD = q$ and $mC = -a\overline{q}/\lambda^3$. Recall also that $m \in \lambda^{-3}\mathcal{O}$, and $C, D \in \mathcal{O}$, and that $(C, D) = 1$. This means $D|\lambda^3 q$. Fix such a D. Then $m = q/D$ and so

$$C = -\frac{a\overline{q}D}{\lambda^3 q} = -\overline{q}\frac{\dfrac{a}{(a, \lambda^3 q)}D}{\dfrac{\lambda^3 q}{(a, \lambda^3 q)}}.$$

It is clear that $a/(a, \lambda^3 q)$ and $\lambda^3 q/(a, \lambda^3 q)$ are relatively prime, and since $C \in \mathcal{O}$, we must therefore have $\lambda^3 q/(a, \lambda^3 q)$ divides D. (Recall that $\overline{q}$ is a root of unity!)

On the other hand, since $(C,D)=1$, and since $a/(a,\lambda^3q)\in\mathcal{O}$, we must also have D divides $\lambda^3q/(a,\lambda^3q)$. Thus we obtain a non-zero contribution only when $D=\lambda^3q/(a,\lambda^3q)$, which implies $m=q/D=(a,\lambda^3q)/\lambda^3$, and $C=-aD/\lambda^3q=-a/(a,\lambda^3q)$, as claimed.

Now choose A,B such that

$$A\lambda^3q+Ba\bar{q}=(a,\lambda^3q).$$

Then

$$\begin{pmatrix} A & B & \\ -a\bar{q}/(a,\lambda^3q) & \lambda^3q/(a,\lambda^3q) & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \\ a\bar{q}/\lambda^3q & 1 & \\ & & 1\end{pmatrix}$$
$$=\begin{pmatrix}1 & B(a,\lambda^3q)/\lambda^3q & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}(a,\lambda^3q)/\lambda^3q & & \\ & \lambda^3q/(a,\lambda^3q) & \\ & & 1\end{pmatrix}.$$

Replace ψ by its Fourier expansion, omit those terms which we have shown to be zero, apply the above formula to the remaining terms, and use (2.2) to get the last statement in Lemma 6A. The Kubota symbol, κ_a, appearing in the statement of the Lemma, comes from the Fourier expansion, cf. [BH2, (1.7)]. As we do not explicitly compute the Fourier coefficients of ψ at the various cusps in this paper, there is no reason to further evaluate this Kubota symbol.

We are now able to compute the poles of the integral I_2 which we defined in the discussion following Lemma 3. The individual Fourier coefficients in (2.4) with $mn\neq 0$ are of rapid decay in y as $y\to 0$ so the partial Mellin transforms of these terms are entire in w. The poles arise from the partial Mellin transforms of the terms

$$\psi^{\rho}_{r,0}\begin{pmatrix}r/yq^3 & & \\ & t/r & \\ & & 1\end{pmatrix}. \tag{2.5}$$

The multiplicity one theorem implies that $\psi^{\rho}_{r,0}(\tau)$ is a constant multiple of $\psi^{1}_{r,0}(\tau)$, and so the pole locations of the partial Mellin transforms of (2.5) are *at most* the same as the pole locations of the partial Mellin transforms of

$$\psi_{r,0}\begin{pmatrix}r/yq^3 & & \\ & t/r & \\ & & 1\end{pmatrix}.$$

(Cancellation could occur in the calculation of the residues contributed by the various terms.)

PROPOSITION 7A: *If $r \in \lambda^{-3}\mathcal{O}$, then*

$$\begin{aligned}\psi_{r,0}(\tau) =& \tau(-r)e(rx_1)y_1^{s+1}y_2^{2s}K_{\frac{1}{3}}(4\pi|r|y_1)\\ &+ e(rx_1)F_2(s)r^{-\frac{5}{3}+\frac{3}{2}s}y_1^{\frac{7}{3}-\frac{s}{2}}y_2^{\frac{8}{3}-s}K_{\frac{3}{2}s-\frac{5}{3}}(4\pi|r|y_1)\end{aligned}$$

where

$$F_2(s) = \frac{V3^5(2\pi)^{\frac{3}{2}s+\frac{1}{3}}}{4(\frac{3}{2}s-\frac{4}{3})\Gamma(\frac{3}{2}s-\frac{2}{3})}\sum_{B\neq 0}\sum_{\substack{A,C(B)\\ d=(A,B)}}\left(\frac{a}{b}\right)\left(\frac{d}{C}\right)e(-rA/B)|d|^{\frac{4}{3}}|B|^{-3s-\frac{2}{3}}.$$

PROPOSITION 7B: *If $r \in \lambda^{-3}\mathcal{O}$, then*

$$\psi_{r,0}(\tau) = e(rx_1)F_3(s)r^{-\frac{5}{3}+\frac{3}{2}s}y_1^{\frac{7}{3}-\frac{s}{2}}y_2^{\frac{8}{3}-s}K_{\frac{3}{2}s-\frac{5}{3}}(4\pi|r|y_1)$$

where

$$F_3(s) = \frac{V3^5(2\pi)^{\frac{3}{2}s+\frac{1}{3}}}{4(\frac{3}{2}s-\frac{4}{3})\Gamma(\frac{3}{2}s-\frac{2}{3})}\sum_{C\equiv 1(3)}\sum_{\substack{A,B(C)\\ d=(A,C)}}\left(\frac{a}{c}\right)\left(\frac{d}{B}\right)e(-rA/C)|d|^{\frac{4}{3}}|C|^{-3s-\frac{2}{3}}.$$

We wish to compute the poles of

$$\int_0^1\int_0^\infty \psi_{r,0}\begin{pmatrix} r/yq^3 & & \\ & t/r & \\ & & 1\end{pmatrix} y^{2w}t^z\frac{dt}{t}\frac{dy}{y}. \tag{2.6}$$

The matrix in the integrand of (2.6) has Iwasawa coordinates $x_1 = 0$, $y_1 = t/r$, and $y_2 = r^2/ytq^3$. Applying Proposition 7A, we see that we actually want to compute the pole locations of

$$\int_0^1\int_0^\infty \Big[\tau(-r)t^{1-s}r^{3s-1}y^{-2s}q^{-6s}K_{\frac{1}{3}}(4\pi t) + F_2(s)t^{-1/3-s/2}y^{s-8/3}r^{4/3}q^{3s-8}K_{\frac{3}{2}s-\frac{5}{3}}(4\pi t)\Big]y^{2w}t^z\frac{dt}{t}\frac{dy}{y}.$$

We obtain poles at $w = s$, and $w = 4/3 - s/2$. Recall that I_3 also had a pole at $w = s - 1$. Recall now the Remark at the beginning of this section. These three poles are the poles of the Dirichlet series

$$\sum_n \frac{L(\frac{3}{2}s-1, \chi_{n^2})}{N(n)^{w-s+1}}.$$

If we set $w' = w - s + 1$, then we have poles at $w' = 1$, $7/3 - 3s/2$, and 0, as desired! This is precisely the first assertion of Main Theorem A. Main Theorem B is similar, except (cf. Lemma 7B) we do not get a pole at $w = s$. We now prove the second assertion in Main Theorem A.

THEOREM 8A: *Suppose q is equivalent to a root of unity mod 3. Using notation as above, we have the functional equation*

$$\int_0^\infty\int_0^\infty\int_{\mathbb{C}}\sum_{n\neq 0}\phi_{1,n}\left(\begin{pmatrix}1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}yt & & \\ xyt & t & \\ & & 1\end{pmatrix}\right)y^{2w}t^z dx\frac{dt}{t}\frac{dy}{y}$$

$$=\frac{1}{VN(\lambda^5q^3)}\sum_{a\in\mathcal{O}/\lambda^5q^2} e\left(-\frac{a\overline{q}\mu'-nB(a,\lambda^3q)}{\lambda^3q}\right)\kappa_a$$

$$\times\int_0^\infty\int_0^\infty\sum_{n\neq 0}\psi^{\rho}_{r,n}\begin{pmatrix}ry/q^3 & & \\ & t/r & \\ & & 1\end{pmatrix}y^{-2w}t^z dx\frac{dt}{t}\frac{dy}{y}.$$

The equality is meant in the sense of analytic continuation.

Theorem 8A follows from Lemma 6A and this observation: suppose $f(y)$ is a linear combination of powers of y and let

$$f_0(w)=\int_0^1 f(y)y^w dy \qquad \text{and} \qquad f_\infty(w)=\int_1^\infty f(y)y^w dy,$$

initially defined for large and small w, respectively. Then the functions f_0 and f_∞ are meromorphic on $\mathbb{C}$, and $f_0=-f_\infty$.

The term

$$\int_{\mathbb{C}}\phi_{1,0}\begin{pmatrix}yt & & \\ xyt & t & \\ & & 1\end{pmatrix}dx$$

is a linear combination of powers of y, so the above observation applies. A similar statement holds for $\psi_{r,0}$. This is essentially the reason that no $n=0$ terms appear in Theorem 8A.

Proof of Theorem 8A: Express the equation in Lemma 6A as

$$\sum_n f_n(y,t)=\sum_n g_n(y^{-1},t). \tag{2.7}$$

We have

$$\int_0^\infty \int_0^\infty \sum_{n\neq 0} f_n(y,t) y^{2w} t^z \frac{dy}{y}\frac{dt}{t}$$

$$= \int_1^\infty \int_0^\infty \sum_{n\neq 0} f_n(y,t) y^{2w} t^z \frac{dy}{y}\frac{dt}{t} + \int_0^1 \int_0^\infty \sum_{n} f_n(y,t) y^{2w} t^z \frac{dy}{y}\frac{dt}{t}$$
$$- \int_0^1 \int_0^\infty f_0(y,t) y^{2w} t^z \frac{dy}{y}\frac{dt}{t}$$

$$(2.8)\qquad = \int_1^\infty \int_0^\infty \sum_{n\neq 0} f_n(y,t) y^{2w} t^z \frac{dy}{y}\frac{dt}{t} + \int_1^\infty \int_0^\infty \sum_{n} g_n(y,t) y^{-2w} t^z \frac{dy}{y}\frac{dt}{t}$$
$$- \int_0^1 \int_0^\infty f_0(y,t) y^{2w} t^z \frac{dy}{y}\frac{dt}{t}$$

$$= \int_1^\infty \int_0^\infty \sum_{n\neq 0} f_n(y,t) y^{2w} t^z \frac{dy}{y}\frac{dt}{t} + \int_1^\infty \int_0^\infty \sum_{n\neq 0} g_n(y,t) y^{-2w} t^z \frac{dy}{y}\frac{dt}{t}$$
$$+ \int_1^\infty \int_0^\infty g_0(y,t) y^{-2w} t^z \frac{dy}{y}\frac{dt}{t} - \int_0^1 \int_0^\infty f_0(y,t) y^{2w} t^z \frac{dy}{y}\frac{dt}{t}.$$

The first two integrals converge for all w, and by Propositions 2 and 7A the second two converge for w large, and have meromorphic continuation to all of $\mathbb{C}$.

Next we have, again by (2.7),

$$\int_0^\infty \int_0^\infty \sum_{n\neq 0} g_n(y,t) y^{-2w} t^z \frac{dy}{y}\frac{dt}{t}$$

$$= \int_1^\infty \int_0^\infty \sum_{n\neq 0} g_n(y,t) y^{-2w} t^z \frac{dy}{y}\frac{dt}{t} + \int_0^1 \int_0^\infty \sum_{n} g_n(y,t) y^{-2w} t^z \frac{dy}{y}\frac{dt}{t}$$

$$- \int_0^1 \int_0^\infty g_0(y,t) y^{-2w} t^z \frac{dy}{y}\frac{dt}{t}$$

$$= \int_1^\infty \int_0^\infty \sum_{n\neq 0} g_n(y,t) y^{-2w} t^z \frac{dy}{y}\frac{dt}{t} + \int_1^\infty \int_0^\infty \sum_{n} f_n(y,t) y^{2w} t^z \frac{dy}{y}\frac{dt}{t}$$

$$(2.9) \qquad - \int_0^1 \int_0^\infty g_0(y,t) y^{-2w} t^z \frac{dy}{y}\frac{dt}{t}$$

$$= \int_1^\infty \int_0^\infty \sum_{n\neq 0} g_n(y,t) y^{-2w} t^z \frac{dy}{y}\frac{dt}{t} + \int_1^\infty \int_0^\infty \sum_{n\neq 0} f_n(y,t) y^{2w} t^z \frac{dy}{y}\frac{dt}{t}$$

$$+ \int_1^\infty \int_0^\infty f_0(y,t) y^{2w} t^z \frac{dy}{y}\frac{dt}{t} - \int_0^1 \int_0^\infty g_0(y,t) y^{-2w} t^z \frac{dy}{y}\frac{dt}{t}$$

The first two integrals converge for all w, and by Propositions 2 and 7A the second two converge for w sufficiently small, and have meromorphic continuation to all $\mathbb{C}$. By the remarks preceding this proof, the continuation of the right side of (2.8) equals the continuation of the right side of (2.9). This completes the proof of Theorem 8A, and so that of Main Theorem A.

We now finish the proof of Main Theorem B. To begin, we now set the notation $\psi(\tau) = \tilde{\phi}(w_2\tau)$, and continue our previous work, beginning at the end of Lemma 5B. We now restrict ourselves to the case where $3|q$ and μ is congruent to a root of unity mod 3. Again, ψ has a Fourier expansion; in this case the contribution of an individual term is just

(2.9)

$$\int_{(\mathbb{C}/3)^2} \psi^{\rho}_{m,n}\left(\begin{pmatrix} A & B & \\ C & D & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & & x \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & & \\ a\overline{\mu}/\lambda^3 q & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} 1/y\overline{q}q^2 & & \\ & t/q & \\ & & 1\end{pmatrix}\right)$$
$$\times\, e\left(-q\xi_1 + \frac{xa\overline{\mu}}{\lambda^3}\right) dx d\xi_1.$$

The following Lemma is proved exactly as is Lemma 6A. We do not include the proof here.

LEMMA 6B: *The integral* (2.9) *is zero except in the case when*

$$m = (a, \lambda^3 q)/\lambda^3,$$
$$C = -a\overline{\mu}/(a, \lambda^3 q),$$
$$D = \lambda^3 q/(a, \lambda^3 q).$$

Remark: Again, we will assume that A and B are chosen so that

$$A\lambda^3 q + Ba\bar{q} = (a, \lambda^3 q).$$

The functional equation is given by the following Theorem. Its proof is identical to the proof of Theorem 8A; we do not include the proof here.

THEOREM 8B: *Using notation as above, we have the functional equation*

$$\int_0^\infty \int_0^\infty \int_{\mathbb{C}} \sum_{n\neq 0} \phi_{1,n}\left(\begin{pmatrix} 1 & \mu/q & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} yt & & \\ xyt & t & \\ & & 1 \end{pmatrix}\right) y^{2w} t^z dx \frac{dt}{t}\frac{dy}{y}$$

$$= \frac{1}{VN(\lambda^5 q^3)} \sum_{a\in\mathcal{O}/\lambda^5 q^2} e\left(-\frac{a\bar{\mu}\mu' - nB(a,\lambda^3 q)}{\lambda^3 q}\right)\kappa_a$$

$$\times \int_0^\infty \int_0^\infty \sum_{n\neq 0} \psi^{\rho}_{r,n}\begin{pmatrix} ry/q^3 & & \\ & t/r & \\ & & 1 \end{pmatrix} y^{-2w} t^z dx \frac{dt}{t}\frac{dy}{y}.$$

This completes the proofs of Main Theorem A and Main Theorem B.

3. The deferred proofs

We now exhibit the deferred proofs.

Proof of Proposition 2: This proof closely follows Section 4 of [BH]. We compute:

$$\phi_{r,0}(\tau) = \phi_{r,0}\left(\begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right)$$

$$= e(rx_1)\phi_{r,0}\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}$$

$$= \frac{e(rx_1)}{V^3}\iiint_{(\mathbb{C}/3)^3} \phi\left(\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right) e(-r\xi_1) d\xi_{1,2,3}$$

$$= \frac{e(rx_1)}{V^3}\iiint_{(\mathbb{C}/3)^3} E_s\left(w_1\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right) e(-r\xi_1) d\xi_{1,2,3}$$

$$= \frac{e(rx_1)}{V^3}\iiint_{(\mathbb{C}/3)^3} \sum_{g:[A,B,C]_1} \kappa(g) I_s\left(gw_1\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}\right)$$

$$\times e(-r\xi_1) d\xi_{1,2,3}$$

where we have introduced the notation $g : [A,B,C]_1$ to indicate a sum over matrices g with invariants $(A,B,C,-pA,-qA,d)$ such that $(A,B,C)=1$, $C\equiv 1(3)$, $A\equiv B\equiv 0(3)$, $d=(B,C)\equiv 1(3)$, $b=B/d$, $c=C/d$, and $pc+qb=1$. (We also have, by abuse of notation, written $d\xi_{1,2,3}$ for $d\xi_1 d\xi_2 d\xi_3$. We will continue these notations throughout the remainder of this paper.) This exactly corresponds to summing over $g\in\Gamma_P(3)\backslash\Gamma(3)$, where the bottom row of g, ${}^\iota g$, is $(A\ B\ C)$, $(-pA\ \ -qA\ d)$, respectively. From [BH (1.5)], we have

$$\begin{aligned}\kappa(g)&=\left(\frac{b}{c}\right)\left(\frac{-qA}{1}\right)\left(\frac{C}{1}\right)^{-1}\left(\frac{A}{d}\right)\left(\frac{-pA}{1}\right)\\&=\left(\frac{b}{c}\right)\left(\frac{A}{d}\right).\end{aligned}$$

Using [BH, (4.3)], and a change of variables, we get

$$\begin{aligned}&\phi_{r,0}(\tau)\\&\quad=\frac{e(rx_1)}{V^3}\sum_{[A,B,C]_1}\left(\frac{b}{c}\right)\left(\frac{A}{d}\right)\\&\qquad\cdot\iiint_{(\mathbb{C}/3)^3} I_s\left(\begin{pmatrix}d^{-1}&&\\&-d/C&\\&&C\end{pmatrix}w_1\begin{pmatrix}1&B/C&A/C\\&1&qA/d\\&&1\end{pmatrix}\begin{pmatrix}1&\xi_2&\xi_3\\&1&\xi_1\\&&1\end{pmatrix}\right.\\&\qquad\qquad\left.\times\begin{pmatrix}y_1y_2&&\\&y_1&\\&&1\end{pmatrix}\right)e(-r\xi_1)d\xi_{1,2,3}\\&\quad=\frac{e(rx_1)}{V^3}\sum_{[A,B,C]_1}\left(\frac{b}{c}\right)\left(\frac{A}{d}\right)e(rqA/d)\\&\qquad\cdot\iiint_{(\mathbb{C}/3)^3} I_s\left(\begin{pmatrix}d^{-1}&&\\&-d/C&\\&&C\end{pmatrix}w_1\begin{pmatrix}1&\xi_2+B/C&\xi_3+A/C\\&1&\xi_1\\&&1\end{pmatrix}\right.\\&\qquad\qquad\left.\times\begin{pmatrix}y_1y_2&&\\&y_1&\\&&1\end{pmatrix}\right)e(-r\xi_1)d\xi_{1,2,3}.\end{aligned}$$

We can let A and B run mod C, completing the integrals on ξ_2 and ξ_3, giving

$$\phi_{r,0}(\tau) = \frac{e(rx_1)}{V^3} \sum_{C\equiv 1(3)} \sum_{\substack{A,B \bmod C \\ (A,B,C)=1 \\ A\equiv B\equiv 0(3)}} \left(\frac{b}{c}\right)\left(\frac{A}{d}\right) e(rqA/d)$$

$$\cdot \int_{\mathbb{C}/3}\int_{\mathbb{C}}\int_{\mathbb{C}} I_s\left(\begin{pmatrix} d^{-1} & & \\ & -d/C & \\ & & C\end{pmatrix} w_1 \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right) \times e(-r\xi_1)d\xi_{1,2,3}.$$

Now,

$$\begin{pmatrix} d^{-1} & & \\ & -d/C & \\ & & C\end{pmatrix} w_1 \begin{pmatrix} 1 & 0 & 0 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix} = w_2 \begin{pmatrix} -d/C & & \\ & d^{-1} & \\ & & C\end{pmatrix}\begin{pmatrix} 1 & \xi_1 & 0 \\ & 1 & 0 \\ & & 1\end{pmatrix} w_4$$

and as in [BH, p.497]

$$\begin{pmatrix} 1 & \xi_1 & 0 \\ & 1 & 0 \\ & & 1\end{pmatrix} w_4 \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & 0 \\ & & 1\end{pmatrix}\begin{pmatrix} y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}$$

has coordinates

$$\begin{pmatrix} 1 & \xi_1+\xi_2' & \xi_3' \\ & 1 & \xi_1' \\ & & 1\end{pmatrix}\begin{pmatrix} y_1'y_2' & & \\ & y_1' & \\ & & 1\end{pmatrix}$$

where

$$\begin{aligned} y_1' &= P^{-1}Q^{\frac{1}{2}}y_1, \\ y_2' &= Q^{-1}P^{\frac{1}{2}}y_2, \\ \xi_2' &= -\overline{\xi}_2\xi_3 Q^{-1}, \\ P &= |\xi_3|^2 + |\xi_2|^2y_1^2 + y_1^2y_2^2, \\ Q &= |\xi_2|^2 + y_2^2. \end{aligned}$$

Using these new coordinates, the left invariance of I_s under $w_2 \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1\end{pmatrix}$, and the definition of I_s, we get

$$\begin{aligned}
&\phi_{r,0}(\tau)\\
&=\frac{e(rx_1)}{V^3}\sum_{C\equiv 1(3)}\sum_{\substack{A,B \bmod C\\(A,B,C)=1\\A\equiv B\equiv 0(3)}}\left(\frac{b}{c}\right)\left(\frac{A}{d}\right)e(rqA/d)\\
&\quad\cdot\int_{\mathbb{C}}\int_{\mathbb{C}}\int_{\mathbb{C}/3} I_s\left(\begin{pmatrix} d/C & & \\ & d^{-1} & \\ & & C\end{pmatrix}\begin{pmatrix}1 & \xi_1+\xi_2' & \xi_3'\\ & 1 & \xi_1'\\ & & 1\end{pmatrix}\begin{pmatrix} y_1'y_2' & & \\ & y_1' & \\ & & 1\end{pmatrix}\right)\\
&\qquad\qquad\times e(-r\xi_1)d\xi_{1,2,3}\\
&=\frac{e(rx_1)}{V^3}\sum_{C\equiv 1(3)}\sum_{\substack{A,B \bmod C\\(A,B,C)=1\\A\equiv B\equiv 0(3)}}\left(\frac{b}{c}\right)\left(\frac{A}{d}\right)e(rqA/d)\\
&\quad\cdot\int_{\mathbb{C}}\int_{\mathbb{C}}\int_{\mathbb{C}/3}\left({y_1'}^2y_2'/C^3\right)^s\theta\begin{pmatrix} y_2'd^2/|C| & (\xi_1+\xi_2')d^2/|C|\\ & 1\end{pmatrix}e(-r\xi_1)d\xi_{1,2,3}.
\end{aligned}$$

Inserting the Fourier expansion of θ, we get a contribution only if $d^2\mu/C=r$, that is, $\mu=rC/d^2\in(\lambda^{-3})$. So,

$$\begin{aligned}
\phi_{r,0}(\tau)=&\frac{e(rx_1)}{V^2}\sum_{C\equiv 1(3)}\sum_{\substack{A,B \bmod C\\(A,B,C)=1\\A\equiv B\equiv 0(3)\\d^2|\lambda^3 rC}}\left(\frac{b}{c}\right)\left(\frac{A}{d}\right)e(rqA/d)\tau(rC/d^2)\frac{d^2}{C^{1+3s}}\\
&\cdot\int_{\mathbb{C}}\int_{\mathbb{C}}\left({y_1'}^2y_2'\right)^s y_2'K_{\frac{1}{3}}\left(4\pi|r|y_2'\right)e(\xi_2')d\xi_{2,3}\\
=&\frac{e(rx_1)}{V^2}F_1(s)\int_{\mathbb{C}}\int_{\mathbb{C}}\left({y_1'}^2y_2'\right)^s y_2'K_{\frac{1}{3}}\left(4\pi|r|y_2'\right)e(\xi_2')d\xi_{2,3},
\end{aligned}$$

say. To complete the proof, substitute the definitions of y_1', y_2', and ξ_2', and change variables $\xi_2\mapsto y_2\overline{\xi_2}$, $\xi_3\mapsto y_1y_2\xi_3$.

Note: The above integral equals

$$\begin{aligned}
&\int_{\mathbb{C}}\int_{\mathbb{C}}\frac{\left(|\xi_3|^2+|\xi_2|^2y_1^2+y_1^2y_2^2\right)^{\frac{1}{2}-\frac{3}{2}s}}{|\xi_2|^2+y_2^2}\\
&\qquad\times K_{\frac{1}{3}}\left(4\pi|r|y_2\frac{\sqrt{|\xi_3|^2+|\xi_2|^2y_1^2+y_1^2y_2^2}}{|\xi_2|^2+y_2^2}\right)e\left(\frac{-\overline{\xi_2}\xi_3}{|\xi_2|^2+y_2^2}\right)d\xi_{2,3}.
\end{aligned}$$

To complete the lemma, we note that $\begin{pmatrix} yt & & \\ xyt & t & \\ & & 1 \end{pmatrix}$ has coordinates

$$y_1 = t\Delta,$$
$$y_2 = y/\Delta^2,$$
$$x_1 = 0,$$

where $\Delta = \sqrt{1+|x|^2y^2}$. Substituting into the above equation and changing variables $x \mapsto x/y$, $t \mapsto t/\Delta$, we get

$$I_3 = \frac{1}{V^2}F_1(s)\int_0^1 y^{2w-2s+2}\frac{dy}{y}\int_{\mathbb{C}}\frac{1}{(1+|x|^2)^{-2-2s+z/2}}dx\int_0^\infty I_1(t,s)t^{3-s+z}\frac{dt}{t}.$$

Explicitly evaluating the first two integrals leads to the given result.

Proof of Lemma 3: The proof will take several steps. We proceed directly through the computation, proving the necessary lemmas as needed.

The first step is merely notational. Let

$$G(\tau) = \phi\left(\begin{pmatrix} 1 & \mu/q & \\ & 1 & \\ & & 1 \end{pmatrix}\tau\right). \tag{3.0}$$

Since ϕ is automorphic under $\Gamma(3)$, it follows easily that

$$G\left(\begin{pmatrix} 1 & & n_3 \\ & 1 & qn_1 \\ & & 1 \end{pmatrix}\tau\right) = G(\tau),$$

for all $n_1, n_3 \in 3\mathcal{O}$. If $M \in \mathbb{Z}$, then the characters of $\mathbb{C}$ which are trivial on $M\mathcal{O}$ are exactly those of the form $\psi(a) = e(ma)$ with $m \in (\lambda M)^{-1}\mathcal{O}$, so this set parametrizes the characters of the group $\mathbb{C}/M\mathcal{O}$. Putting $M = 3 = -\lambda^2$ gives the Fourier expansion

$$G(\tau) = \sum_{n_1,n_3\in\lambda^{-3}\mathcal{O}} G_{n_1}^{n_3}(\tau),$$

where

$$G_{n_1}^{n_3}(\tau) = \frac{1}{V^2N(q)}\int_{\mathbb{C}/3}\int_{\mathbb{C}/3q} G\left(\begin{pmatrix} 1 & & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\tau\right) e\left(-\frac{n_1}{q}\xi_1 - n_3\xi_3\right) d\xi_{1,3}.$$

Note that the definition of $G_{n_1}^{n_3}(\tau)$ is not completely analogous to that of $\phi_{n_1}^{n_3}(\tau)$. The notation involving G is local to this Lemma and should not cause any confusion.

LEMMA:

$$\phi_1^0\left(\begin{pmatrix}1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\tau\right) = G_q^0(\tau).$$

Proof: We compute:

$$\begin{aligned}
&\phi_1^0\left(\begin{pmatrix}1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\tau\right)\\
=&\frac{1}{V^2}\int\limits_{\mathbb{C}/3}\int\limits_{\mathbb{C}/3}\phi\left(\begin{pmatrix}1 & & \xi_3\\ & 1 & \xi_1\\ & & 1\end{pmatrix}\begin{pmatrix}1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\tau\right)e(-\xi_1)d\xi_{1,3}\\
=&\frac{1}{V^2}\int\limits_{\mathbb{C}/3}\int\limits_{\mathbb{C}/3}\phi\left(\begin{pmatrix}1 & \mu/q & \xi_3\\ & 1 & \xi_1\\ & & 1\end{pmatrix}\tau\right)e(-\xi_1)d\xi_{1,3}\\
=&\frac{1}{V^2}\int\limits_{\mathbb{C}/3}\int\limits_{\mathbb{C}/3}\phi\left(\begin{pmatrix}1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1 & & \xi_3-\xi_1\mu/q\\ & 1 & \xi_1\\ & & 1\end{pmatrix}\tau\right)e(-\xi_1)d\xi_{1,3}\\
=&\frac{1}{V^2}\int\limits_{\mathbb{C}/3}\int\limits_{\mathbb{C}/3}G\left(\begin{pmatrix}1 & & \xi_3\\ & 1 & \xi_1\\ & & 1\end{pmatrix}\tau\right)e(-\xi_1)d\xi_{1,3}\\
=&\frac{1}{V^2N(q)}\int\limits_{\mathbb{C}/3}\int\limits_{\mathbb{C}/3q}G\left(\begin{pmatrix}1 & & \xi_3\\ & 1 & \xi_1\\ & & 1\end{pmatrix}\tau\right)e(-\xi_1)d\xi_{1,3}\\
=&G_q^0(\tau),
\end{aligned}$$

as claimed.

Now we begin the calculation. Write $I(\tau,\mu/q)$ for the left side of the lemma.

We have

$$I(\tau,\mu/q) = \int_{\mathbb{C}} G_q^0\left(\begin{pmatrix}1&&\\x&1&\\&&1\end{pmatrix}\tau\right)dx$$
$$= \sum_{m\in\mathcal{O}}\int_{\mathbb{C}/3q^2} G_q^0\left(\begin{pmatrix}1&&\\x+3mq^2&1&\\&&1\end{pmatrix}\tau\right)dx \tag{3.1}$$
$$= \sum_{m\in\mathcal{O}}\int_{\mathbb{C}/3q^2} G_q^0\left(\begin{pmatrix}1&&\\3mq^2&1&\\&&1\end{pmatrix}\begin{pmatrix}1&&\\x&1&\\&&1\end{pmatrix}\tau\right)dx$$
$$= \sum_{m\in\mathcal{O}}\int_{\mathbb{C}/3q^2} G_q^{3mq^2}\left(\begin{pmatrix}1&&\\x&1&\\&&1\end{pmatrix}\tau\right)dx.$$

Only the last step is unclear. It follows from this

LEMMA: *If $m\in\mathcal{O}$ then*

$$G_q^0\left(\begin{pmatrix}1&&\\3mq^2&1&\\&&1\end{pmatrix}\tau\right) = G_q^{3mq^2}(\tau).$$

Proof: We compute:

$$V^2N(q)G_q^0\left(\begin{pmatrix}1&&\\mq^2&1&\\&&1\end{pmatrix}\tau\right)$$
$$= \int_{\mathbb{C}/3}\int_{\mathbb{C}/3q} G\left(\begin{pmatrix}1&&\xi_3\\&1&\xi_1\\&&1\end{pmatrix}\begin{pmatrix}1&&\\mq^2&1&\\&&1\end{pmatrix}\tau\right)e(-\xi_1)d\xi_{1,3}$$
$$= \int_{\mathbb{C}/3}\int_{\mathbb{C}/3q} \phi\left(\begin{pmatrix}1&\mu/q&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&&\xi_3\\mq^2&1&\xi_1\\&&1\end{pmatrix}\tau\right)e(-\xi_1)d\xi_{1,3}$$
$$= \int_{\mathbb{C}/3}\int_{\mathbb{C}/3q} \phi\left(\begin{pmatrix}1&\mu/q&\\&1&\\&&1\end{pmatrix}\begin{pmatrix}1&&\\mq^2&1&\\&&1\end{pmatrix}\begin{pmatrix}1&&\xi_3\\&1&\xi_1-mq^2\xi_3\\&&1\end{pmatrix}\tau\right)$$
$$\times e(-\xi_1)d\xi_{1,3}$$

$$= \int_{\mathbb{C}/3}\int_{\mathbb{C}/3q} \phi\left(\begin{pmatrix} 1+\mu mq & -\mu^2 m & \\ mq^2 & 1-\mu mq & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\tau\right) \times e(-\xi_1 - mq^2\xi_3)d\xi_{1,3}$$

$$= \int_{\mathbb{C}/3}\int_{\mathbb{C}/3q} \phi\left(\begin{pmatrix} 1 & \mu/q & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\tau\right) e(-\xi_1 - mq^2\xi_3)d\xi_{1,3}$$

$$= \int_{\mathbb{C}/3}\int_{\mathbb{C}/3q} G\left(\begin{pmatrix} 1 & & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\tau\right) e(-\xi_1 - mq^2\xi_3)d\xi_{1,3}$$

$$= V^2 N(q) G_q^{mq^2}(\tau).$$

For the fifth equality we used the automorphy of ϕ under $\Gamma(3)$; this requires $m \in 3\mathcal{O}$. The other equalities are clear, proving the lemma.

For the next step we will replace the sum of $G_q^{3mq^2}(\tau)$ by a sum of $G_q^m(\tau)$. Suppose $m \in \lambda^{-1}\mathcal{O}$. Then $\psi(a) = e\left(\frac{m}{\lambda^3 q^2}a\right)$ is a character of $\mathcal{O}/\lambda^5 q^2$, which is trivial only if $m \in \lambda^2 q^2\mathcal{O} = 3q^2\mathcal{O}$. Thus,

$$\sum_{a\in\mathcal{O}/\lambda^5 q^2} e\left(\frac{m}{\lambda^3 q^2}a\right) = \begin{cases} N(\lambda^5 q^2), & \text{if } m \in 3q^2\mathcal{O}, \\ 0, & \text{otherwise.}\end{cases}$$

It follows easily from the definition that

$$G_n^m\left(\begin{pmatrix} 1 & & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\tau\right) = e\left(\frac{n}{q}\xi_1 + m\xi_3\right) G_n^m(\tau), \tag{3.2}$$

so we have

$$\sum_{a\in\mathcal{O}/\lambda^5 q^2}\ \sum_{m\in\lambda^{-3}\mathcal{O}} G_q^m\left(\begin{pmatrix} 1 & & a/\lambda^3 q^2 \\ & 1 & \\ & & 1\end{pmatrix}\tau\right) = N(\lambda^5 q^2)\sum_{m\in\mathcal{O}} G_q^{3q^2 m}(\tau). \tag{3.3}$$

Combining (3.1) and (3.3) we get

$$I(\tau,\mu/q) = \frac{1}{N(\lambda^5 q^2)}\sum_{a\in\mathcal{O}/\lambda^5 q^2}\int_{\mathbb{C}/3q^2}\sum_{m\in\lambda^{-3}\mathcal{O}} G_q^m\left(\begin{pmatrix} 1 & & a/\lambda^3 q^2 \\ x & 1 & \\ & & 1\end{pmatrix}\tau\right) dx. \tag{3.4}$$

Now Lemma 2 follows immediately from (3.4) and the following

LEMMA:

$$\sum_{n_3\in\lambda^{-3}\mathcal{O}} G_q^{n_3}(\tau) = \frac{1}{VN(q)} \int_{\mathbb{C}/3q} G\left(\begin{pmatrix}1 & & \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\tau\right) e(-\xi_1)d\xi_1.$$

Proof: We compute:

$$\begin{aligned}
&\int_{\mathbb{C}/3q} G\left(\begin{pmatrix}1 & & \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\tau\right) e(-\xi_1)d\xi_1 \\
&= \int_{\mathbb{C}/3q} \sum_{n_1,n_3\in\lambda^{-3}\mathcal{O}} G_{n_1}^{n_3}\left(\begin{pmatrix}1 & & \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\tau\right) e(-\xi_1)d\xi_1 \\
&= \int_{\mathbb{C}/3q} \sum_{n_1,n_3\in\lambda^{-3}\mathcal{O}} G_{n_1}^{n_3}(\tau) e\left(\frac{n_1}{q}\xi_1 - \xi_1\right) d\xi_1 \\
&= VN(q) \sum_{n_3\in\lambda^{-3}\mathcal{O}} G_q^{n_3}(\tau).
\end{aligned}$$

The first step is just the Fourier expansion of $G(\tau)$, the second used (3.2), and the third follows because we are integrating a nontrivial character unless $n_1/q = 1$.

Combine (3.4), (3.0), and the above Lemma to finish the proof of Lemma 3.

Proof of Lemma 4A: By definition, $\phi(\tau) = \tilde{\phi}({}^\iota\tau)$, so all we do is apply this relation to the left side of the equation in the Lemma. The rest of the proof involves using the automorphic properties of ϕ to put things in the appropriate form. Write $\phi(*)$ for the left side of the Lemma. First note that

$$\begin{pmatrix}yt & & \\ xyt & t & \xi_1 \\ & & 1\end{pmatrix} = \begin{pmatrix} & & -1 \\ 1 & & \\ & -1 & \end{pmatrix}\begin{pmatrix}t & -\xi_1 & -xyt \\ & 1 & \\ & & yt\end{pmatrix}\begin{pmatrix} & 1 & \\ & & -1 \\ -1 & & \end{pmatrix}.$$

Using this equation and then applying the involution we have

$$\begin{aligned}
\phi(*) &= \phi\left(\begin{pmatrix}1 & \mu/q & a/\lambda^3q^2 \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} & & -1 \\ 1 & & \\ & -1 & \end{pmatrix}\begin{pmatrix}t & -\xi_1 & -xyt \\ & 1 & \\ & & yt\end{pmatrix}\right) \\
&= \tilde{\phi}\left({}^\iota\begin{pmatrix}1 & \mu/q & a/\lambda^3q^2 \\ & 1 & \\ & & 1\end{pmatrix} {}^\iota\begin{pmatrix} & & -1 \\ 1 & & \\ & -1 & \end{pmatrix} {}^\iota\begin{pmatrix}t & -\xi_1 & -xyt \\ & 1 & \\ & & yt\end{pmatrix}\right)
\end{aligned}$$

$$=\tilde{\phi}\left(\begin{pmatrix}1 & & -a/\lambda^3q^2\\ & 1 & -\mu/q\\ & & 1\end{pmatrix}\begin{pmatrix} & -1 & \\ & & 1\\ -1 & & \end{pmatrix}\begin{pmatrix}1/y & & x\\ & t & \xi_1\\ & & 1\end{pmatrix}\right)$$

$$=\tilde{\phi}\left(w_4\begin{pmatrix}1 & & \\ -a/\lambda^3q^2 & 1 & \\ -\mu/q & & 1\end{pmatrix}\begin{pmatrix}1/y & & x\\ & t & \xi_1\\ & & 1\end{pmatrix}\right).$$

If $\gamma\in\Gamma(3)$ then $\tilde{\phi}(w_4\gamma\tau)=\tilde{\phi}(w_4\tau)$ because ϕ, hence $\tilde{\phi}$, is invariant under $\Gamma(3)$, and $\Gamma(3)$ is a normal subgroup of $\Gamma(1)$. We use this to fix up the matrices inside $\tilde{\phi}$. Choose μ' and d as in the statement of the Lemma. Then,

$$\begin{pmatrix}d & & \mu'\\ & 1 & \\ \mu\overline{q} & & q\overline{q}\end{pmatrix}\in\Gamma(3),$$

so we have

$$\phi(*)=\tilde{\phi}\left(w_4\begin{pmatrix}d & & \mu'\\ & 1 & \\ \mu\overline{q} & & q\overline{q}\end{pmatrix}\begin{pmatrix}1 & & \\ a/\lambda^3q^2 & 1 & \\ -\mu/q & & 1\end{pmatrix}\begin{pmatrix}1/y & & x\\ & t & \xi_1\\ & & 1\end{pmatrix}\right)$$

$$=\tilde{\phi}\left(w_4\begin{pmatrix}1/q\overline{q} & & \mu'\\ a/\lambda^3q^2 & 1 & \\ & & q\end{pmatrix}\begin{pmatrix}1/y & & x\\ & t & \xi_1\\ & & 1\end{pmatrix}\right)$$

$$=\tilde{\phi}\left(w_4\begin{pmatrix}1 & & \mu'/q\\ a\overline{q}/\lambda^3q & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1/q\overline{q}y & & x/q\overline{q}\\ & t & \xi_1\\ & & q\end{pmatrix}\right),$$

as claimed.

Proof of Lemma 4B: As in the proof of Lemma 4A, we may use the definition $\phi(\tau)=\tilde{\phi}({}^{\iota}\tau)$ to obtain (writing $\phi(*)$ for the left side of the Lemma)

$$\phi(*)=\tilde{\phi}\left(w_4\begin{pmatrix}1 & & \\ -a/\lambda^3q^2 & 1 & \\ -\mu/q & & 1\end{pmatrix}\begin{pmatrix}1/y & & x\\ & t & \xi_1\\ & & 1\end{pmatrix}\right)$$

$$=\tilde{\phi}\left(\begin{pmatrix}a/\lambda^3q & -1 & 0\\ \mu/q & & 1\\ -1 & & \end{pmatrix}\begin{pmatrix}1/y & & x\\ & t & \xi_1\\ & & 1\end{pmatrix}\right).$$

Choose μ' and d as in the statement of the Lemma. Then,

$$\begin{pmatrix}1 & & \\ & \mu' & d\\ & q\overline{\mu} & \mu\overline{\mu}\end{pmatrix}\in\Gamma(3),$$

so we have

$$\begin{aligned}\phi(*) &= \tilde{\phi}\left(\begin{pmatrix}1 & & \\ & \mu' & d \\ & q\bar{\mu} & \mu\bar{\mu}\end{pmatrix}\begin{pmatrix}a/\lambda^3 q & -1 & 0 \\ \mu/q & & 1 \\ -1 & & \end{pmatrix}\begin{pmatrix}1/y & & x \\ & t & \xi_1 \\ & & 1\end{pmatrix}\right)\\
&= \tilde{\phi}\left(\begin{pmatrix}a/\lambda^3 q & -1 & \\ 1/q\bar{\mu} & & \mu' \\ & & q\bar{\mu}\end{pmatrix}\begin{pmatrix}1/y & & x \\ & t & \xi_1 \\ & & 1\end{pmatrix}\right)\\
&= \tilde{\phi}\left(w_2\begin{pmatrix}1 & & \mu'/q \\ a\bar{\mu}/\lambda^3 q & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}1/q\bar{\mu}y & & x/q\bar{\mu} \\ & t & \xi_1 \\ & & q\end{pmatrix}\right),\end{aligned}$$

as desired.

Proof of Proposition 7A: We compute:

$$\begin{aligned}
&V^3\, {}^{w_4}\tilde{\phi}_{r,0}(\tau)\\
&= \iiint_{(\mathbb{C}/3)^3} \tilde{\phi}\left(w_4\begin{pmatrix}1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1\end{pmatrix}\right) e(-r\xi_1)d\xi_{1,2,3}\\
&= e(rx_1)\iiint_{(\mathbb{C}/3)^3} \tilde{\phi}\left(w_4\begin{pmatrix}1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right) e(-r\xi_1)d\xi_{1,2,3}\\
&= e(rx_1)\iiint_{(\mathbb{C}/3)^3} \phi\left({}^{\iota}w_4\, {}^{\iota}\!\begin{pmatrix}1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix} {}^{\iota}\!\begin{pmatrix}y_1y_2 & & \\ & y_1 & \\ & & 1\end{pmatrix}\right) e(-r\xi_1)d\xi_{1,2,3}\\
&= e(rx_1)\iiint_{(\mathbb{C}/3)^3} \phi\left(w_5\begin{pmatrix}1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & \\ & y_2 & \\ & & 1\end{pmatrix}\right) e(r\xi_2)d\xi_{1,2,3}\\
&= e(rx_1)\iiint_{(\mathbb{C}/3)^3} E_s\left(w_2\begin{pmatrix}1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & \\ & y_2 & \\ & & 1\end{pmatrix}\right) e(r\xi_2)d\xi_{1,2,3}\\
&= e(rx_1)\\
&\quad\times\iiint_{(\mathbb{C}/3)^3} \sum_{g:[A,B,C]} \kappa(g) I_s\left(gw_2\begin{pmatrix}1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & \\ & y_2 & \\ & & 1\end{pmatrix}\right) e(r\xi_2)d\xi_{1,2,3}\\
&= e(rx_1)\,(X_1+X_2+X_3)\,,
\end{aligned}$$

say, where the last three terms come from the three cases described below. The sum $g : [A, B, C]$ is over $g \in \Gamma_P(3)\backslash\Gamma(3)$, with $(A\ B\ C)$ the bottom row of g. The integers A, B, C satisfy $(A,B,C)=1$, $3|(A,B)$, and $C\equiv 1(3)$. This specifies the coset of g, but not the specific matrix g. Our method of choosing g depends on which of A, B, C are nonzero; this results in the three terms X_1, X_2, and X_3. Specifically, the bottom row of gw_2 is $(B\ A\ C)$, and we will write gw_2 in terms of its Bruhat decomposition. There are three cases to consider:

CASE 1: $B\neq 0$. Choose g to have invariants $(A,B,C,d,-qC,-pC)$ with $d=(A,B)$, $a=A/d$, $b=B/d$, and $pa+qb=1$, with $p\equiv -1(3)$, $q\equiv 0(3)$. Then gw_2 has invariants $(B,A,C,d,-pC,-qC)$, so by Proposition 3.7 of [BFG],

$$gw_2=\begin{pmatrix}1 & -(a_{21}B-a_{11}A)/d & a_{11}/b\\ & 1 & a_{21}/B\\ & & 1\end{pmatrix} w_1\begin{pmatrix}B & & \\ & -d/B & \\ & & 1/d\end{pmatrix}\begin{pmatrix}1 & A/B & C/B\\ & 1 & -pC/d\\ & & 1\end{pmatrix}.$$

Since $d|a_{21}B-a_{11}A$, we have

$$X_1=\iiint_{(\mathbb{C}/3)^3}\sum_{\substack{g:[A,B,C]\\ B\neq 0\\ 3|B}}\kappa(g)I_s\left(w_1\begin{pmatrix}B & & \\ & -d/B & \\ & & 1/d\end{pmatrix}\begin{pmatrix}1 & A/B & C/B\\ & 1 & pC/d\\ & & 1\end{pmatrix}\right.$$
$$\left.\times\begin{pmatrix}1 & \xi_2 & \xi_3\\ & 1 & \xi_1\\ & & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & \\ & y_2 & \\ & & 1\end{pmatrix}\right)e(r\xi_2)d\xi_{1,2,3}.$$

By (1.5) in [BH],

$$\kappa(g)=\left(\frac{a}{b}\right)\left(\frac{d}{C}\right).$$

Since $\kappa(g)$ depends only on A and C mod B, we let A,C run mod B, and thus complete the integrals on ξ_2 and ξ_3, giving

$$X_1=\sum_{\substack{B\neq 0\\ 3|B}}\sum_{A,C(B)}\left(\frac{a}{b}\right)\left(\frac{d}{C}\right)e(-rA/B)$$
$$\cdot\int_{\mathbb{C}}\int_{\mathbb{C}}\int_{\mathbb{C}/3}I_s\left(w_1\begin{pmatrix}B & & \\ & -d/B & \\ & & 1/d\end{pmatrix}\begin{pmatrix}1 & \xi_2 & \xi_3\\ & 1 & \xi_1\\ & & 1\end{pmatrix}\begin{pmatrix}y_1y_2 & & \\ & y_2 & \\ & & 1\end{pmatrix}\right)$$
$$\times e(r\xi_2)d\xi_{1,2,3}$$

$$= \sum_{\substack{B\neq 0\\ 3|B}} \sum_{A,C(B)} \left(\frac{a}{b}\right)\left(\frac{d}{C}\right) e(-rA/B)$$

$$\cdot \int_{\mathbb{C}}\int_{\mathbb{C}}\int_{\mathbb{C}/3} I_s\left(S\begin{pmatrix} d/B & & \\ & 1/d & \\ & & B\end{pmatrix} w_4 \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\begin{pmatrix} y_1y_2 & & \\ & y_2 & \\ & & 1\end{pmatrix}\right)$$

$$\times e(r\xi_2)d\xi_{1,2,3}$$

where

$$S = \begin{pmatrix} & -1 & \\ 1 & & \\ & & 1\end{pmatrix}.$$

Note that $I_s(S\tau) = I_s(\tau)$. Now we manipulate some matrices, in preparation for applying the definition of I_s. First,

$$w_4\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1\end{pmatrix} = w_4 \begin{pmatrix} 1 & & \\ & 1 & \xi_1 \\ & & 1\end{pmatrix}\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \\ & & 1\end{pmatrix}$$
$$= \begin{pmatrix} 1 & \xi_1 & \\ & 1 & \\ & & 1\end{pmatrix} w_4 \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \\ & & 1\end{pmatrix};$$

and by (4.6) of [BH],

$$w_4\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} y_1y_2 & & \\ & y_2 & \\ & & 1\end{pmatrix} = \begin{pmatrix} 1 & \xi_2' & \xi_3' \\ & 1 & \xi_1' \\ & & 1\end{pmatrix}\begin{pmatrix} y_1'y_2' & & \\ & y_1' & \\ & & 1\end{pmatrix}$$

where

$$y_1' = y_2P^{-1}Q^{\frac{1}{2}},$$
$$y_2' = y_1Q^{-1}P^{\frac{1}{2}},$$
$$\xi_2' = -\overline{\xi}_2\xi_3Q^{-1},$$
$$P = |\xi_3|^2 + y_2^2|\xi_2|^2 + y_1^2y_2^2,$$
$$Q = |\xi_2|^2 + y_1^2.$$

Now we use the above calculations and insert the definition of I_s:

$$X_1 = \sum_{\substack{B\neq 0\\ 3|B}} \sum_{\substack{A,C(B)\\ d=(A,B)}} \left(\frac{a}{b}\right)\left(\frac{d}{C}\right) e(-rA/B)$$

$$\cdot \int_{\mathbb{C}}\int_{\mathbb{C}}\int_{\mathbb{C}/3} I_s\left(\begin{pmatrix} d/B & & \\ & 1/d & \\ & & B\end{pmatrix}\begin{pmatrix} 1 & \xi_1+\xi_2' & * \\ & 1 & * \\ & & 1\end{pmatrix}\begin{pmatrix} y_1'y_2' & & \\ & y_1' & \\ & & 1\end{pmatrix}\right)$$

$$\times e(r\xi_2)d\xi_{1,2,3}$$

$$= \sum_{\substack{B\neq 0\\ 3|B}} \sum_{A,C(B)} \left(\frac{a}{b}\right)\left(\frac{d}{C}\right) e(-rA/B)$$

$$\cdot \int_{\mathbb{C}}\int_{\mathbb{C}}\int_{\mathbb{C}/3} (|B|^{-3}{y_1'}^2 y_2')^s \theta\left(\begin{pmatrix} d^2/B & \\ & 1\end{pmatrix}\begin{pmatrix} y_2' & \xi_1+\xi_2' \\ & 1\end{pmatrix}\right) e(r\xi_2)d\xi_{1,2,3}.$$

From (1.14) in [BH] we have the Fourier expansion

$$\theta\begin{pmatrix} y & x \\ & 1\end{pmatrix} = \frac{3^{\frac{5}{2}}}{2} y^{\frac{2}{3}} + y \sum_{\mu\in\lambda^{-3}} \tau(\mu) K_{\frac{1}{3}}(4\pi|\mu|y) e(\mu x).$$

Inserting this expansion, we see that the integral over ξ_1 vanishes for all but the "constant term." Plugging in the definitions of y_1' and y_2' gives

$$X_1 = \frac{3^5 V}{2} \sum_{B\neq 0} \sum_{A,C(B)} \left(\frac{a}{b}\right)\left(\frac{d}{C}\right) e(-rA/B)|d|^{\frac{4}{3}}|B|^{-3s-\frac{2}{3}}$$

$$\cdot \int_{\mathbb{C}}\int_{\mathbb{C}} {y_2'}^{\frac{2}{3}} \left({y_1'}^2 y_2'\right)^s e(r\xi_2)d\xi_{2,3}$$

$$= \frac{3^5 V}{2} \sum_{B\neq 0} \sum_{A,C(B)} \left(\frac{a}{b}\right)\left(\frac{d}{C}\right) e(-rA/B)|d|^{\frac{4}{3}}|B|^{-3s-\frac{2}{3}}$$

$$\cdot y_1^{\frac{2}{3}+s} y_2^{2s} \int_{\mathbb{C}}\int_{\mathbb{C}} (|\xi_3|^2 + y_2^2|\xi_2|^2 + y_1^2y_2^2)^{\frac{1}{3}-\frac{3}{2}s}(|\xi_2|^2+y_1^2)^{-\frac{2}{3}} e(r\xi_2)d\xi_{2,3}.$$

The contribution of Case 1 is completed with this lemma.

LEMMA:

$$\int_{\mathbb{C}}\int_{\mathbb{C}} (|\xi_3|^2 + y_2^2|\xi_2|^2 + y_1^2y_2^2)^{\frac{1}{3}-\frac{3}{2}s}(|\xi_2|^2+y_1^2)^{-\frac{2}{3}} e(r\xi_2)d\xi_{2,3}$$

$$= \frac{\pi(2\pi)^{\frac{3}{2}s-\frac{2}{3}} r^{-\frac{5}{3}+\frac{3}{2}s}}{\left(\frac{3}{2}s-\frac{4}{3}\right)\Gamma\left(\frac{3}{2}s-\frac{2}{3}\right)} y_1^{\frac{5}{3}-\frac{3}{2}s} y_2^{\frac{8}{3}-3s} K_{\frac{3}{2}s-\frac{5}{3}}(4\pi|r|y_1).$$

Proof: We apply formulas (3.2) and (3.3) in [BH]. By (3.2),

$$\int_{\mathbb{C}} (|\xi_3|^2 + y_2^2|\xi_2|^2 + y_1^2y_2^2)^{\frac{1}{3}-\frac{3}{2}s} d\xi_3 = \frac{\pi}{\frac{3}{2}s-\frac{4}{3}} \left(y_2^2|\xi_2|^2 + y_1^2y_2^2\right)^{\frac{4}{3}-\frac{3}{2}s}.$$

Writing $I(y_1, y_2)$ for the left side of the Lemma, applying the above formula, and then applying (3.3) we get

$$\begin{aligned} I(y_1,y_2) &= \frac{\pi}{\frac{3}{2}s-\frac{4}{3}} \int\limits_{\mathbb{C}} \left(y_2^2|\xi_2|^2 + y_1^2y_2^2\right)^{\frac{4}{3}-\frac{3}{2}s} \left(|\xi_2|^2+y_1^2\right)^{-\frac{2}{3}} e(r\xi_2)d\xi_2 \\ &= \frac{\pi y_2^{\frac{8}{3}-3s}}{\frac{3}{2}s-\frac{4}{3}} \int\limits_{\mathbb{C}} \left(|\xi_2|^2+y_1^2\right)^{\frac{2}{3}-\frac{3}{2}s} e(r\xi_2)d\xi_2 \\ &= \frac{\pi y_2^{\frac{8}{3}-3s} r^{-\frac{10}{3}+3s}}{\frac{3}{2}s-\frac{4}{3}} \int\limits_{\mathbb{C}} \left(|\xi_2|^2+|r|^2y_1^2\right)^{\frac{2}{3}-\frac{3}{2}s} e(\xi_2)d\xi_2 \\ &= \frac{\pi(2\pi)^{\frac{3}{2}s-\frac{2}{3}} r^{-\frac{5}{3}+\frac{3}{2}s}}{\left(\frac{3}{2}s-\frac{4}{3}\right)\Gamma\left(\frac{3}{2}s-\frac{2}{3}\right)} y_1^{\frac{5}{3}-\frac{3}{2}s} y_2^{\frac{8}{3}-3s} K_{\frac{3}{2}s-\frac{5}{3}}\left(4\pi|r|y_1\right), \end{aligned}$$

as claimed.

CASE 2: $B=0$, $A\neq 0$. We shall see that there is no no contribution from this cell. Choose g to have invariants $(A,0,C,-A,0,C)$, so gw_2 has invariants $(0,A,C,-A,C,0)$. By Proposition 3.5 of [BFG],

$$gw_2 = \begin{pmatrix} 1 & -a_{11} & * \\ & 1 & 0 \\ & & 1 \end{pmatrix} w_5 \begin{pmatrix} -1 & & \\ & A & \\ & & -1/A \end{pmatrix} \begin{pmatrix} 1 & -a_{22} & -a_{23} \\ & 1 & C/A \\ & & 1 \end{pmatrix}$$

and by (1.5) in [BH],

$$\kappa(g) = \left(\frac{A}{C}\right),$$

so we have

$$\begin{aligned} X_2 = \sum_{\substack{A,C \\ A\neq 0}} \iiint\limits_{(\mathbb{C}/3)^3} I_s\Bigg(w_5 \begin{pmatrix} -1 & & \\ & A & \\ & & -1/A \end{pmatrix} \\ \times \begin{pmatrix} 1 & -a_{22} & -a_{23} \\ & 1 & C/A \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2 & & \\ & y_2 & \\ & & 1 \end{pmatrix} \Bigg) e(r\xi_2)d\xi_{1,2,3} \\ = \sum_{\substack{A,C \\ A\neq 0}} \iiint\limits_{(\mathbb{C}/3)^3} I_s\Bigg(w_5 \begin{pmatrix} -1 & & \\ & A & \\ & & -1/A \end{pmatrix} \\ \times \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 + C/A \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2 & & \\ & y_2 & \\ & & 1 \end{pmatrix} \Bigg) e(r\xi_2)d\xi_{1,2,3} \end{aligned}$$

$$= \sum_{\substack{A\neq 0\\ C(A)}} \int_{\mathbb{C}/3}\int_{\mathbb{C}/3}\int_{\mathbb{C}} I_s\Bigg(w_5 \begin{pmatrix} -1 & & \\ & A & \\ & & -1/A \end{pmatrix} \times \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & & \\ & y_2 & \\ & & 1 \end{pmatrix}\Bigg) e(r\xi_2)d\xi_{1,2,3}.$$

We have the matrix identity:

$$w_5 \begin{pmatrix} -1 & & \\ & A & \\ & & -1/A \end{pmatrix}\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} = S\begin{pmatrix} -1 & & \\ & -1/A & \\ & & A \end{pmatrix}\begin{pmatrix} 1 & \xi_3 & \\ & 1 & \\ & & 1 \end{pmatrix} w_3 \begin{pmatrix} 1 & \xi_2 & \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix};$$

and $w_3 \begin{pmatrix} y_1y_2 & y_2\xi_2 & \\ & y_2 & \xi_1 \\ & & 1 \end{pmatrix}$ has coordinates $\begin{pmatrix} y_1y_2 & y_2\xi_2 & \\ & y_2/\Delta & \overline{\xi_1}/\Delta \\ & & \Delta \end{pmatrix}$ where $\Delta^2 = y_2^2 + |\xi_1|^2$. So

$$X_2 = \sum_{\substack{A\neq 0\\ C(A)}} \left(\frac{A}{C}\right)$$

$$\int_{\mathbb{C}/3}\int_{\mathbb{C}/3}\int_{\mathbb{C}} I_s\left(\begin{pmatrix} -1 & & \\ & -1/A & \\ & & A \end{pmatrix}\begin{pmatrix} y_1y_2 & y_2\xi_2 & \\ & y_2/\Delta & \overline{\xi_1}/\Delta \\ & & \Delta \end{pmatrix}\right) e(r\xi_2)d\xi_{1,2,3}.$$

Replace $y_2\xi_2 + y_2\xi_3/\Delta$ by ξ_3, and note that the integral over ξ_2 vanishes, provided $r \in \lambda^{-3}\mathcal{O}$.

CASE 3: $A = B = 0$, $C = 1$. Choose g so that gw_2 has coordinates $(0,0,1,0,0,1)$. Then,

$$\begin{aligned} X_3 &= \iiint_{(\mathbb{C}/3)^3} I_s\left(\begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix}\begin{pmatrix} y_1y_2 & & \\ & y_2 & \\ & & 1 \end{pmatrix}\right) e(r\xi_2)d\xi_{1,2,3} \\ &= \iiint_{(\mathbb{C}/3)^3} (y_1y_2^2)^s \theta\begin{pmatrix} y_1 & \xi_2 \\ & 1 \end{pmatrix} e(r\xi_2)d\xi_{1,2,3} \\ &= V^3\tau(-r)y_1^{s+1}y_2^{2s}K_{\frac{1}{3}}(4\pi|r|y_1), \end{aligned}$$

by term-by-term integration of the Fourier expansion of θ.

Adding together the three cases gives Proposition 7A.

Proof of Proposition 7B: Exactly mimicing the proof of Proposition 7A, we get

$$\begin{aligned} &V^{3\ w_2}\tilde{\phi}_{r,0}(\tau) \\ &= \iiint_{(\mathbb{C}/3)^3} \tilde{\phi}\left(w_2 \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2 & y_1x_2 & x_3 \\ & y_1 & x_1 \\ & & 1 \end{pmatrix}\right) e(-r\xi_1) d\xi_{1,2,3} \\ &= e(rx_1) \iiint_{(\mathbb{C}/3)^3} \sum_{g:[A,B,C]} \kappa(g) I_s\left(gw_2 \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2 & & \\ & y_2 & \\ & & 1 \end{pmatrix}\right) \\ &\qquad \times e(r\xi_2) d\xi_{1,2,3} \\ &= X, \end{aligned}$$

say. The sum $g : [A, B, C]$ is over $g \in \Gamma_P(3)\backslash\Gamma(3)$, with $(A\ B\ C)$ the bottom row of g. The integers A, B, C satisfy $(A, B, C) = 1$, $3|(A, B)$, and $C \equiv 1(3)$. This specifies the coset of g, but not the specific matrix g. We choose g to have invariants $(A, B, C, qB - A, d, C - pb)$, where $d = (A, C)$, $a = A/d$, $b = B/d$, and $pa - qc = 1$. By (1.5) of [BH] this gives

$$\kappa(g) = \left(\frac{a}{c}\right)\left(\frac{d}{B}\right).$$

Proceeding exactly as in Case 1 of the proof of Proposition 7A, we first have

$$\begin{aligned} gw_4 = &\begin{pmatrix} 1 & -(a_{21}C - a_{11}A)/d & * \\ & 1 & * \\ & & 1 \end{pmatrix} \\ &\times w_1 \begin{pmatrix} C & & \\ & -d/C & \\ & & 1/d \end{pmatrix} \begin{pmatrix} 1 & A/C & B/C \\ & 1 & (C-pB)/d \\ & & 1 \end{pmatrix}. \end{aligned}$$

Since $d|a_{21}C - a_{11}A$, we have

$$\begin{aligned} X = &\iiint_{(\mathbb{C}/3)^3} \sum_{g:[A,B,C]} \left(\frac{a}{c}\right)\left(\frac{d}{B}\right) I_s\left(w_1 \begin{pmatrix} C & & \\ & -d/C & \\ & & 1/d \end{pmatrix} \right. \\ &\times \left. \begin{pmatrix} 1 & A/C & B/C \\ & 1 & (C-pB)/d \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2 & & \\ & y_2 & \\ & & 1 \end{pmatrix}\right) e(r\xi_2) d\xi_{1,2,3}. \end{aligned}$$

We can complete the integrals on ξ_2 and ξ_3, giving

$$X = \sum_{C\equiv 1(3)} \sum_{A,B(C)} \left(\frac{a}{c}\right)\left(\frac{d}{B}\right) e(-rA/B)$$

$$\int_{\mathbb{C}}\int_{\mathbb{C}}\int_{\mathbb{C}/3} I_s\left(w_1 \begin{pmatrix} B & & \\ & -d/B & \\ & & 1/d \end{pmatrix} \begin{pmatrix} 1 & \xi_2 & \xi_3 \\ & 1 & \xi_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1y_2 & & \\ & y_2 & \\ & & 1 \end{pmatrix}\right)$$

$$\times\ e(r\xi_2)d\xi_{1,2,3}.$$

The above integrals were evaluated in the proof of Proposition 7A. This finishes the proof of Proposition 7B.

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